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STABILITY OF THE DISPLACEMENT OF IMMISCIBLE VISCO-ELASTIC LIQUIDS IN A POROUS MEDIUM†

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The stability of the motion of the boundary between two visco-elastic liquids in a porous medium caused by the non-equilibrium of filtration fluxes is investigated. Two stages in the development of the instability are considered: the first is the development of small perturbations on the surface of the initially unperturbed displacement front and the second is the evolution of small perturbations on the surfaces of “fingers” of displacing liquid extruded after the development of the first-stage instability. The theoretical analysis shows that the use of visco-elastic liquids for displacing oil from strata should increase the stability of the displacement process. In the case of visco-elastic liquids with relaxation along the pressure gradient the stability of the displacement is due to stabilization of the actual boundary between the liquids, whereas with relaxation along the flux it is achieved because of the instability of the “fingers” of displacing fluid.

DURING the displacement of liquids of greater viscosity by less viscous liquids or gases there is a viscous instability of the displacement front in a porous medium. Much work (see, e.g. the reviews

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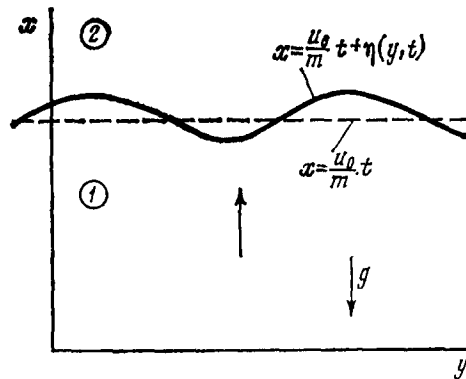


FIG. 1.

[1, 2]) has been devoted to the study of the appearance of this instability and the development of the “fingers” of displacing fluid associated with it. Experimental investigations of the displacement of visco-elastic fluids in a porous medium have revealed a range of qualitatively new properties [3, 4]. One of them is a tendency towards an increased stability of the displacement front, even for unfavourable mobility ratios. Experiments in a Hele–Shaw cell have shown that under identical conditions replacing the displacing viscous fluid by a visco-elastic fluid can lead to stabilization of the displacement front [3]. The possibility of reducing the oil-output coefficient as the temperature increases has been demonstrated [4], and this is explained by the weakening of the visco-elastic properties of the oil being displaced.

1. STABILITY OF THE DISPLACEMENT FRONT

Consider the constant-velocity vertical motion of a plane boundary between two visco-elastic fluids with differing rheological characteristics (Fig. 1). The distribution of the filtration velocities and pressures has the form

$$u_{j0} = u_0, \quad v_{j0} = 0, \quad p_{j0} = p_0 - (K_j^{-1}u_0 + \rho_j g) (x - m^{-1}u_0 t) \\ K_j = k_j / \mu_j, \quad j = 1, 2$$

where m is the porosity and u_j , p_j , μ_j and k_j are the filtration velocity, pressure, viscosity and the permeability of the j th phase; the indices $j = 1$ and $j = 2$ correspond to the displacing and displaced liquids, respectively.

The system of linearized equations for the two-dimensional filtration of visco-elastic liquids, describing the evolution of a perturbation of the main flow, includes the continuity and filtration equations [5]:

$$\frac{\partial u_j'}{\partial x} + \frac{\partial v_j'}{\partial y} = 0, \quad R_j u_j' = -K_j Q_j \frac{\partial p_j'}{\partial x}, \quad R_j v_j' = -K_j Q_j \frac{\partial p_j'}{\partial y} \quad (1.1) \\ u_j' = u_j - u_{j0}, \quad v_j' = v_j - v_{j0}, \quad p_j' = p_j - p_{j0} \\ R_j = 1 + \tau_j \left(\frac{\partial}{\partial t} + \frac{u_0}{m} \frac{\partial}{\partial x} \right), \quad Q_j = 1 + \theta_j \left(\frac{\partial}{\partial t} + \frac{u_0}{m} \frac{\partial}{\partial x} \right)$$

(\mathbf{R} , and \mathbf{Q} , are differential operators in the linear approximation and τ_j and θ_j are relaxation times).

The equation of the separation boundary of the liquids $x - m^{-1}u_0t = \eta(y, t)$ is related to the velocity field by the kinematics relations

$$m\partial\eta/\partial t - u_j' = 0 \quad (1.2)$$

The dynamical condition of pressure continuity across the separation boundary gives

$$p_1' - p_2' = [(\rho_1 - \rho_2)g + (K_1^{-1} - K_2^{-1})u_0]\eta \quad (1.3)$$

It follows from relations (1.1) that the pressure perturbations satisfy Laplace's equation. Hence the solution to problem (1.1)–(1.3) can be conveniently represented in the form

$$\begin{aligned} \eta &= A \exp(\delta t + i\kappa y) \\ p_j' &= B_j \exp \psi_j, \quad u_j' = C_j \exp \psi_j, \quad v_j' = D_j \exp \psi_j \\ \psi_j &= \delta t + i\kappa y - (-1)^j \kappa (x - m^{-1}u_0t) \end{aligned}$$

where κ is the wave number and δ is the growth increment of the perturbations. Substitution of these expressions into (1.1)–(1.3) leads to the dispersion equation

$$\begin{aligned} \delta \left(K_1^{-1} \frac{1 + \tau_1\delta}{1 + \theta_1\delta} + K_2^{-1} \frac{1 + \tau_2\delta}{1 + \theta_2\delta} \right) + \frac{\kappa}{m} N &= 0 \\ N &= (K_1^{-1} - K_2^{-1})u_0 + (\rho_1 - \rho_2)g \end{aligned} \quad (1.4)$$

In the case of the filtration of two viscous Newtonian liquids, Eq. (1.4) gives an expression for the growth increment [6]

$$\delta_0 = -N\kappa / [m(K_1^{-1} + K_2^{-1})]$$

from which it follows that the displacement process is only stable when $N > 0$. The presence of relaxational properties in the liquids leads to the possibility that stable displacement can also occur in the case when $N < 0$, i.e. when the mobility of the displacing liquid is greater than that of the displaced liquid.

We will first consider the special case when $\tau_1 = \tau_2 = 0$, for which Eq. (1.4) can be rewritten in the form

$$\begin{aligned} a\delta^2 + b\delta + c &= 0 \\ a &= K_1^{-1}\theta_2 + K_2^{-1}\theta_1 + m^{-1}\kappa N\theta_1\theta_2 \\ b &= K_1^{-1} + K_2^{-1} + m^{-1}\kappa(\theta_1 + \theta_2)N, \quad c = m^{-1}\kappa N \end{aligned} \quad (1.5)$$

Obviously, if $N > 0$, all the coefficients of Eq. (1.5) are positive. If the roots of this equation are real, then they are both negative, and if they are complex, then their real part is negative and the position of the front is stable. If $N < 0$ ($c < 0$), three cases are possible:

1. $a > 0, b > 0$; one of the roots (δ_+) is positive, which corresponds to an unstable displacement front;
2. $a < 0, b > 0$; if the roots of the equation are real, then they are both positive, but if they are complex, their real part is positive and the position of the front is unstable;
3. $a < 0, b > 0$; this case is similar to the case $N > 0$, when all the coefficients of Eq. (1.5) are positive, and reduces to it via the replacements $a \rightarrow -a, b \rightarrow, c \rightarrow -c$; the position of the front is stable.

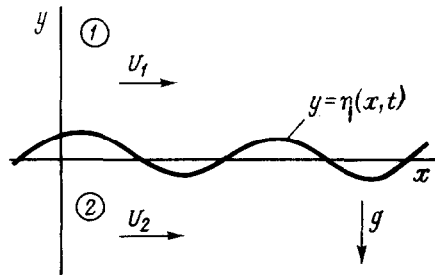


FIG. 2.

One can show that in terms of wave numbers the stability criterion has the form

$$\kappa > \kappa_c = m (K_1^{-1}\theta_2 + K_2^{-1}\theta_1) / [\theta_1\theta_2 (-N)] \quad (1.6)$$

A complete analysis of the roots of Eq. (1.4) shows that the special case considered above is unique. Hence, a stable displacement regime with $N < 0$ is only possible when there is relaxation along the pressure gradient in both liquids ($\theta_1\theta_2 \neq 0$) and there is no relaxation along the fluxes ($\tau_1 = \tau_2 = 0$). The corresponding stability criterion has the form (1.6). Estimates show that for liquids with relaxation times $10^2 c < \theta < 10^4 c$ and with $\mu \approx 0.3$, $k_1\mu_2/(k_2\mu_1) \approx 2$, $\rho_1 \approx \rho_2$, $u_0 \approx 10^{-5}$ m/s the critical wavelength ($\lambda_c = 2\pi/k_c$) varies within the limits $1 \text{ cm} < \lambda_c < 1 \text{ m}$.

2. STABILITY OF "FINGERS" OF DISPLACING LIQUID

The developed instability stage of the displacement is characterized by the existence of a deformed front, accompanied by the break-out of displacing liquid in the form of individual "fingers". Hence, the further solution of the displacement stability problem is bound up with the investigation of the shear stability of the filtrating fluxes.

We will consider two incompressible visco-elastic liquids of different densities, moving in a porous medium with different, though constant, velocities (Fig. 2). The filtration velocity and pressure distributions for the unperturbed flow have the form

$$u_{j0} = U_j, \quad v_{j0} = 0, \quad p_{j0} = p_0 - K_j^{-1}U_j x - \rho_j g y, \quad j = 1, 2 \\ (K_1^{-1}U_1 = K_2^{-1}U_2)$$

The evolution of perturbations of the main flow is described by system (1.1). The kinematic and dynamic conditions on the separation boundary of the liquid $y = \eta(x, t)$ reduce to the relations

$$m \frac{\partial \eta}{\partial t} + U_j \frac{\partial \eta}{\partial x} - v_j' = 0, \quad j = 1, 2; \quad p_1' - p_2' = (\rho_1 - \rho_2) g \eta \quad (2.1)$$

Investigation of the shear stability of parallel filtrating fluxes reduces to an analysis of the stability of the elementary wave packet

$$\eta = A \exp(-i\omega t + i\kappa x) \\ p_j' = B_j \exp \psi_j, \quad u_j' = C_j \exp \psi_j, \quad v_j' = D_j \exp \psi_j \\ \psi_j = -i\omega t + i\kappa x + (-1)^j \kappa y, \quad j = 1, 2$$

The corresponding dispersion equation from problem (1.1), (2.1) can be written in the form

$$\begin{aligned} & K_1 K_2 g (\rho_2 - \rho_1) \kappa F_1 (\omega) F_2 (\omega) - \\ & - i K_1 m \Omega_2 F_1 (\omega) G_2 (\omega) - i K_2 m \Omega_1 F_2 (\omega) G_1 (\omega) = 0 \end{aligned} \quad (2.2)$$

$$F_j (\omega) = 1 - i \theta_j \Omega_j, \quad G_j (\omega) = 1 - i \tau_j \Omega_j, \quad \Omega_j = \omega - m^{-1} U_j \kappa, \quad j = 1, 2$$

In the special case of the filtration of two viscous Newtonian liquids, the solution of Eq. (2.2) can be written explicitly:

$$\begin{aligned} \omega_0 &= v_f x - i \delta_0 \\ v_f &= \frac{K_2 U_1 + K_1 U_2}{m (K_1 + K_2)}, \quad \delta_0 = - \frac{K_1 K_2 g (\rho_2 - \rho_1)}{m (K_1 + K_2)} \kappa \end{aligned} \quad (2.3)$$

From this it follows that the phase velocity of propagation v_f of the perturbations along the separation boundary is constant, and that its growth increment δ_0 does not depend on the velocities of the filtration fluxes; when $\rho_1 > \rho_2$ the Rayleigh–Taylor instability occurs.

Estimates performed for the values $\rho_2 = \rho_1 \approx -10^2 \text{ kg/m}^3$, $k_j \approx 0.2 \times 10^{-12} \text{ m}^2$ and $\mu_j \approx 10^{-3} \text{ kg/(m s)}$ show that the characteristic time for the development of instability for a wave with $\lambda \approx 10^{-2} \text{ m}$ is approximately 1 hour, while for $\lambda \approx 1 \text{ m}$ it is 100 hours.

In Sec. 1 it was shown that extrusion of the displacing liquid in the form of individual “fingers” is most probable when there is no relaxation along the pressure gradients and relaxation occurs along the fluxes. In this case ($\theta_1 = \theta_2 = 0$, $\tau_1 \tau_2 \neq 0$), Eq. (2.2) is rewritten in the form

$$K_2 \tau_1 \Omega_1^2 + K_1 \tau_2 \Omega_2^2 - K_1 K_2 g (\rho_2 - \rho_1) m^{-1} \kappa = i (K_1 \Omega_2 + K_2 \Omega_1). \quad (2.4)$$

For small relaxation times ($\tau_j \omega_j \ll 1$) the perturbation growth increment is computed from the formulae

$$\begin{aligned} \delta &= \delta_0 + \frac{K_1 K_2 \kappa^2}{(K_1 + K_2)^3} [V^2 (\tau_1 K_1 + \tau_2 K_2) - K_1 K_2 m^{-2} g^2 (\rho_2 - \rho_1) (\tau_1 K_2 + \tau_2 K_1)], \\ V &= \frac{U_2 - U_1}{m} \end{aligned}$$

From this it is clear that for small differences in the liquid densities ($\rho_1/\rho_2 \approx 1$) the growth increment δ is significantly positive and the side surface of the “fingers” is unstable.

Under the conditions $\tau_j \Omega_j \gg 1$, Eq. (2.4) reduces to the form

$$\begin{aligned} Z_0 (\Omega_1, \kappa) &= i F \\ Z_0 &\equiv a \Omega_2^2 + \Omega_1^2 - g b (1 - s) \kappa, \quad a = \tau_2 K_1 / (\tau_1 K_2) \\ F &\equiv -K_1 (K_2^{-1} \Omega_2 + K_1^{-1} \Omega_1) / \tau_1, \quad b = K_1 \rho_2 / (m \tau_1), \quad s = \rho_1 / \rho_2 \end{aligned} \quad (2.5)$$

We note that the equation $Z_0 (\Omega_1, \kappa) = 0$ is similar to the dispersion relation describing shear instability in a two-layered perfect liquid [7]. One can show that in this case in the short-wave domain

$$\kappa > \kappa_* = (1 + a) b g (1 - s) / (a V^2)$$

there is a Kelvin–Helmholtz type instability. Waves with $\kappa < \kappa_*$ are neutrally stable.

It is known that for shear flow of a two-layered viscous incompressible liquid there are so-called

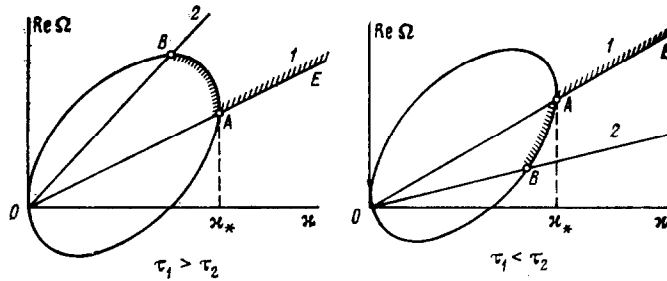


FIG. 3.

“negative energy waves” (see review [8]) which are neutrally stable when there is no viscosity, while in the case of viscosity they lose stability. (Negative energy waves were studied in [9] in the hydrodynamics of visco-elastic liquids.) Proceeding as in the case of two-layered liquid hydrodynamics, we analyse the influence of the right-hand side of Eq. (2.5) on neutrally stable waves. We obtain the expression

$$\delta = - \frac{K_1}{2\tau_1} \frac{(K_1^{-1} + K_2^{-1}) \Omega_0(\kappa) - K_2^{-1} V \kappa}{(a + 1) \Omega_0(\kappa) - a V \kappa} \tag{2.6}$$

for the growth increment.

Here $\Omega_0(\kappa)$ is the root of the equation $Z_0(\Omega_0, \kappa) = 0$.

The sign of δ depends on the relative positions of the lines

$$\Omega_1^{(1)} = \frac{a}{a + 1} V \kappa \quad \text{and} \quad \Omega_1^{(2)} = \frac{a}{a + \tau_2/\tau_1} V \kappa$$

in the (Ω_1, κ) plane; along these lines the denominator and numerator of expression (2.6), respectively vanish.

Figure 3 illustrates the two possible situations ($\tau_1 > \tau_2$) and ($\tau_1 < \tau_2$). The hatching shows the domains of instability (the line AE corresponds to Kelvin–Helmholtz type instability and the line AB to an instability similar to the dissipative instability of “negative energy waves”). Thus, in this case, when the displacement front is unstable ($\theta_1 = \theta_2 = 0$), the surface of the “fingers” of extruded liquid also has a tendency to instability because of the non-equilibrium of the filtrating fluxes.

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